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A NUMERICAL TREATMENT OF TWO-DIMENSIONAL QUADRATIC VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

S. A. RAAD[1] & M. M. ALATAWI[2]

[1]*Department of Mathematics, College of Applied Sciences, Umm Al-Qura University, Saudi Arabia*

[2]*Department of Mathematics, College of Applied Sciences, Tabuk University, Saudi Arabia*

ABSTRACT

In this paper, we considered a nonlinear two-dimensional quadratic Volterra integral equation of the second kind with continuous kernel. The existence of a unique solution of this equation under certain conditions, was discussed in the Banach space C ([0, T] × [0, T]) using Pickard method. Adomian method was applied on the Two-dimensional quadratic Volterra integral equation for a nonlinear case. Two applications were presented; Maple18 software was used to compute the numerical solutions. Finally, we realized that the Adomian method is appropriate and effective for treating two-dimensional linear and nonlinear quadratic integral equations. Moreover, we noted that the effect of time factor was obvious on numerical results.

KEYWORDS: Volterra Integral Equation, Two Dimensional Integral Equations, Quadratic Integral Equations & Adomian Decomposition Method

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Original Article

1. INTRODUCTION

Quadratic integral equations (QIEs) include several types of integral equations. QIEs are often applicable to the theory of neutron transport, traffic theory, modelling of radiative transfer, the queuing theory, the kinetic theory of gases to name a few amongst many other phenomena.

Chandrasekhar equation is an important QIE. This equation describes a radiative transfer. An important special case of that equation is Chandrasekhar's integral equation (CIE) which appears in radiative transfer. There are three areas where CIE is applied which includes stellar atmosphere, the kinetic theory of gases and neutron transport, see Chandrasekhar [1,2], Argyros [3], Caballero et al., [4], Hu et al., [5], and Cahlon [6].

A number of researchers have an interest in QIEs. Concerns lie majorly in numerous studies on the analytical properties of quadratic integral solutions; the properties of QIEs include compactness, the existence of solutions for several classes, monotomic and positive solutions. Abbas et al., [7], Hashem and Alhejelan [8], Darwish [9] and Banas et al., [10] used the theory of measures of noncompactness to prove the existence of theorems for QIEs. Wang et al., [11] studied the existence of positive solutions for the NQIE. El-Sayed and Hashem [12] presented an existence theorem for at least one continuous solution for NQIEs of fractional order. Darwish and Ntouyas [13] presented an existence theorem for monotonic solutions. Zhu [14], using the theory of measures of noncompactness.

Some numerical and analytical methods can be applied to estimate the solutions of the QIEs. However, ADM is the most common method used to obtain the numerical solutions for linear and nonlinear QIEs. In El-

Sayed et al., [15] the Picard and ADM methods were applied in QIEs. Abdou and Raad [16], Raad [17] used the modified ADM, with the aid of modified Simpson's rule (MSR).

2. TWO-DIMENSIONAL VOLTERRA QUADRATIC INTEGRAL EQUATION OF THE SECOND KIND

Consider the T-DVQIE of the second kind

$$\mu\phi(t,\tau) = f(t,\tau) + \chi\big(t,\tau,\phi(t,\tau)\big)\int_0^t\int_0^\tau k(t,r;\tau,s)\,\gamma\big(r,s,\phi(r,s)\big)drds. \tag{1}$$

In (1) μ is a constant and $\mu \neq 0$, the functions $\gamma(\,r,s,\phi(\,r,s))$ and $\big(t,\tau,\phi(t,\tau)\big)$,, are known as continuous nonlinear functions. $k(t,r;\tau,s)$ represents the kernel of the integral equation which will be continuous and the function $f(t,\tau)$ is a known linear continuous function, while $\phi(\,t,\tau)$ is unknown, will be determined in the space$C([0,T]\times[0,T])$.

3. EXISTENCE OF A UNIQUE SOLUTION OF T-DVQIE

In this section, the existence of a unique solution of equation (1) will be discussed and proved using Picard theorem. Considering this, we assume the following conditions:

- The kernel $k(t,r;\tau,s)\epsilon\ C([0,T]\times[0,T])$is a continuous function and satisfies:

  $|k(t,r;\tau,s)| \leq L,\quad (\,H$ is constant).

- The given function $f\ (t,\ \tau)$ and its partial derivatives with respect to $t,\ \tau$ are continuous in $C([0,T]\times[0,T])$.

- The nonlinear function $\gamma\big(t,\tau,\phi(t,\tau)\big)$ is continuous and satisfies for the constants $A_1,\ A_2$ the conditions:

  $(a)\ \big|\gamma\big(t,\tau,\phi(t,\tau)\big)\big| \leq\ A_1.$

  $(b)\big|\ \gamma\big(t,\tau,\phi_1(t,\tau)\big)-\gamma\big(t,\tau,\phi_2(t,\tau)\big)\big|\leq A_2|\phi_1(t,\tau)-\phi_2(t,\tau)|$.

- For the constants B_1, B_2, The nonlinear function $\chi\big(t,\tau,\phi(t,\tau)\big)$ is continuous and satisfies the conditions:

  $(a)\ \big|\chi\big(t,\tau,\phi(t,\tau)\big)\big| \leq\ B_1.$

  $(b)\big|\ \chi\big(t,\tau,\phi_1(t,\tau)\big)-\chi\big(t,\tau,\phi_2(t,\tau)\big)\big|\leq B_2|\phi_1(t,\tau)-\phi_2(t,\tau)|$.

Theorem

The solution of (1) exists and is unique in the Banach space $C([0,T]\times[0,T])$, under the condition $|\mu| > L\ [A_2B_1\ + A_1B_2]$.

Proof

Using Picard method, by picking up any real continuous function $\phi_0(t,\ \tau)$ in the space$C([0,T]\times[0,T])$, then constructing the solution as a sequence of functions $\{\phi_n(t,\tau)\}$,$n \to \infty$.

For every ϕ_n, equation (1), will be in the form

$$\mu\,\phi_n(t,\ \tau) =\ f\ (t,\ \tau) + \chi\big(t,\tau,\phi_{n-1}(t,\tau)\big)\int_0^t\int_0^\tau k(t,r;\tau,s)\,\gamma\big(r,s,\phi_{n-1}(r,s)\big)dsdr,$$

$$n\ =\ 1,\quad 2,\quad 3,\quad ...$$

$$\mu\,\phi_0(t,\ \tau) = f\,(t,\ \tau), \tag{2}$$

functions $\phi_n(t,\tau)$ are continuous and can be written as a sum of differences:

$$\phi_n = \phi_0 + \sum_{j=1}^{n}(\phi_j - \phi_{j-1}) \tag{3}$$

This means that the convergence of the sequence ϕ_n is equivalent to convergence of the infinite series and the solution will be

$$\sum_{j=1}^{n}(\phi_j - \phi_{j-1})$$

$$\phi(t,\ \tau) = \lim_{n\to\infty}\phi_n(t,\ \tau)\,, \quad (\forall\, t,\tau\ \in [0,T]) \tag{4}$$

From (2) for $n\ =\ 1$, we get

$$|\phi_1(t,\ \tau) - \phi_0(t,\ \tau)| = \left|\frac{1}{\mu}\right| \left|\chi(t,\tau,\phi_0(t,\tau))\right| \int_0^t\int_0^\tau |k(t,r;\tau,s)|\left|\gamma(r,s,\phi_0(r,s))\right| dsdr,$$

using assumptions (i), (iii $- a$) and (iv $- a$)

$$|\phi_1(t,\ \tau) - \ \phi_0(t,\ \tau)| \le \left|\frac{1}{\mu}\right| L\,A_1B_1 \int_0^t\int_0^\tau dsdr \le \left|\frac{1}{\mu}\right| L\,A_1B_1\;t\,\tau. \tag{5}$$

Now, we shall obtain an estimate for $n > 1$, from (2) we can get

$$\mu\,[\phi_n(t,\tau) - \phi_{n-1}(t,\tau)] =$$
$$\chi(t,\tau,\phi_{n-1}(t,\tau) - \phi_{n-2}(t,\tau)) \int_0^t\int_0^\tau k(t,r;\tau,s)\gamma(r,s,\phi_{n-1}(r,s) - \phi_{n-2}(r,s))\,dsdr, \tag{6}$$

taking the absolute value of equation (6), to obtain

$$|\phi_n(t,\ \tau) - \phi_{n-1}(t,\ \tau)| = \left|\frac{1}{\mu}\right| \left|\chi(t,\tau,\phi_{n-1}(t,\tau))\right| \int_0^t\int_0^\tau |k(t,r;\tau,s)|\left|\gamma(r,s,\phi_{n-1}(r,s))\right| drds$$
$$- \left|\frac{1}{\mu}\right| \left|\chi(t,\tau,\phi_{n-2}(t,\tau))\right| \int_0^t\int_0^\tau |k(t,r;\tau,s)|\left|\gamma(r,s,\phi_{n-2}(r,s))\right| drds. \tag{7}$$

When $n = 2$, we have

$$|\phi_2(t,\tau) - \phi_1(t,\tau)| \le \left|\frac{1}{\mu}\right|^2 L^2 A_1B_1t^2\tau^2 \left[\frac{1}{4}A_2B_1\ + A_1B_2\right]$$

repeating this technique, we obtain the general form for the terms of the series

$$|\phi_n(t,\tau) - \phi_{n-1}(t,\tau)| \le \left(\left|\frac{1}{\mu}\right| L\;t\;\tau\right)^n A_1B_1 \left[\frac{1}{4}A_2B_1\ + A_1B_2\right]$$
$$\times \left[\frac{1}{9}A_2B_1\ + A_1B_2\right] \ldots \left[\frac{1}{(n-1)^2}A_2B_1\ + A_1B_2\right]\left[\frac{1}{n^2}A_2B_1\ + A_1B_2\right]$$
$$\le \left(\left|\frac{1}{\mu}\right| L\;t\;\tau\,[A_2B_1\ + A_1B_2]\right)^n \le \rho^n, \qquad \left(\rho = \left|\frac{1}{\mu}\right| L\,[A_2B_1\ + A_1B_2]\right). \tag{8}$$

According to the condition ($L\,[A_2B_1\ + A_1B_2] < |\mu|$), when $n \to \infty$, the sequence $\{\phi_j(t,\tau)\}$ is uniformly convergent, and

$$\phi(t,\tau) = \sum_{i=0}^{\infty}\phi_i(t,\ \tau). \tag{9}$$

Since each of $\phi_i(t,\ \tau)$ in (9) is continuous, therefore $\phi(t,\tau)$ is also continuous.

Thus, the existence of a continuous solution of equation (1) is proved.

To prove $\phi(t,\tau)$ is unique, assume $\varphi(t,\tau)$ is another continuous solution, i.e.

$$\mu\,\varphi(t,\ \tau) = f\,(t,\ \tau) + \chi\big(t,\tau,\varphi(t,\tau)\big)\int_0^t\int_0^\tau k(t,r;\tau,s)\,\gamma\big(r,s,\varphi(r,s)\big)dsdr, \tag{10}$$

hence, we get

$$|\varphi(t,\tau)-\phi_n(t,\tau)| \le \left|\frac{1}{\mu}\right| |\chi(t,\tau,\varphi)| \int_0^t\int_0^\tau |k(t,r;\tau,s)||\gamma(r,s,\varphi)-\gamma(r,s,\phi_{n-1})|drds$$

$$+\left|\frac{1}{\mu}\right| |\chi(t,\tau,\varphi)-\chi(t,\tau,\phi_{n-1})| \int_0^t\int_0^\tau |k(t,r;\tau,s)||\gamma(r,s,\phi_{n-1})|drds.$$

Using conditions (iii$-a$) and (iv$-a$), with the aid of equation, we get

$$|\varphi(t,\tau)-\phi_n(t,\tau)| \le \rho^n,\ \ \rho<1, \tag{11}$$

hence

$$\lim_{n\to\infty}\phi_n(t,\ \tau) = \varphi(t,\ \tau) = \phi(t,\ \tau), \qquad (\forall\ t,\tau\ \in[0,T]).$$

which prove that the solution $\phi(t,\ \tau)$ of (1) is unique.

4. ADOMIAN DECOMPOSITION METHOD FOR TWO DIMENSIONAL NONLINEAR VOLTERRA QUADRATIC INTEGRAL EQUATION

ADM has been known as a reliable scheme for solving a variety of linear and nonlinear integral equations. In many papers ADM has been used to solve some classes of integral equations in one dimension. The Adomian polynomials guarantee the convergence of the solution. The convergence of ADM is discussed and proved by different methods.

Here, the T-DNVQIE (1) will be solved using ADM. For this, we express the solution of (1) in the series form by

$$\phi(t,\tau) = \sum_{i=0}^{+\infty}\phi_i\,(t,\tau), \tag{12}$$

the decomposition method identifies the nonlinear terms $\gamma\big(t,\tau,\phi(t,\tau)\big)$, $\chi\big(t,\tau,\phi(t,\tau)\big)$ by

$$\gamma\big(t,\tau,\phi(t,\tau)\big) = \sum_{n=0}^{\infty}A_n\,,\ \ \chi\big(t,\tau,\phi(t,\tau)\big) = \sum_{m=0}^{\infty}\bar{A}_m, \tag{13}$$

where the Adomian polynomials A_n, $\bar{A}_m$ can be determined by the relation:

$$A_n = \frac{1}{n!}\left(\frac{d^n}{d\rho^n}\sum_{i=0}^{\infty}\rho^i\phi(t,\tau)\right)_{\rho=0},\ \ \bar{A}_m = \frac{1}{m!}\left(\frac{d^m}{d\rho^m}\sum_{i=0}^{\infty}\rho^i\phi(t,\tau)\right)_{\rho=0}. \tag{14}$$

Application of ADM to equation. (1) yields

$$\mu\phi_0(t,\tau) = f(t,\tau);$$

$$\mu\phi_i(t,\tau) = f(t,\tau) + \bar{A}_m\int_0^t\int_0^\tau k(t,r;\tau,s)\,A_n\,drds\,,\ \ (i\ge 1). \tag{15}$$

5. NUMERICAL RESULTS

Consider the T-DQVIE (1), when μ = 1 ,, in three different times. The time interval will be divided by $n = m = 10$. The values in the tables are mentioned only in t = τ. The following tables give an exact solution, the approximate solution (Num. Sol.), and their corresponding errors (Error) of both linear and nonlinear types of $\gamma(r, s, \phi(r, s))$. The diagrams explain the difference between these results.

Application (1)

Consider the T-DQVIE

$$\phi(t,\tau) = f(t,\tau) + \frac{1}{8}[\phi^3(t,\tau)] \int_0^t \int_0^\tau (2r + t + s - \tau)\, \phi^k(s,r)\, drds. \tag{16}$$

This application was solved for the linear type when $k = 1$, and $k = 2$ for the nonlinear type. We took three-time values: T = 0.067, 0.1, 0.45.

The exact solution is $\phi(t,\tau) = \sin\left(\frac{\pi}{2}(t+\tau)\right)$

Table 1: $\mathbf{t = \tau = 0.067}$

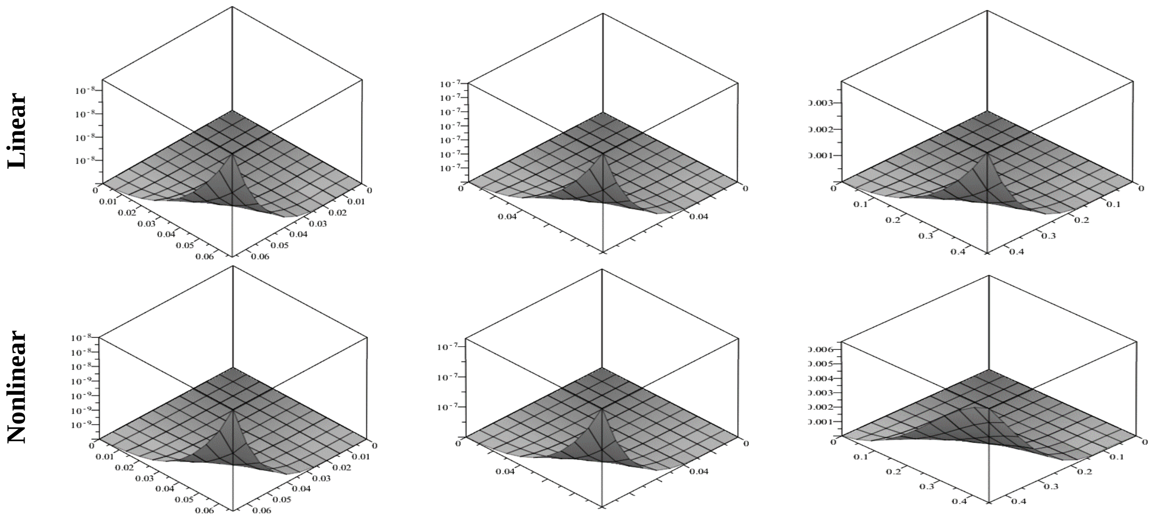
T	τ	Exact	Linear		Nonlinear	
			Num. Sol.	Error	Num. Sol.	Error
0	0	0	0	0	0	0
0.0134	0.0134	0.042084909	0.042084908	0	0.042084909	0
0.0268	0.0268	0.084095246	0.084095246	6.000E-11	0.084095246	1.000E-11
0.402	0.402	0.125956573	0.12595657	1.000E-09	0.125956573	3.000E-10
0.0536	0.0536	0.167594714	0.16759472	9.500E-09	0.167594717	2.400E-09
0.067	0.067	0.20893589	0.208935934	4.460E-08	0.208935904	1.400E-08

Table 2: $\mathbf{t = \tau = 0.1}$

t	τ	Exact	Linear		Nonlinear	
			Num. Sol.	Error	Num. Sol.	Error
0	0	0	0	0	0	0
0.02	0.02	0.06279052	0.06279052	1.000E-11	0.06279052	0
0.04	0.04	0.125333234	0.12533323	1.300E-09	0.125333234	3.000E-10
0.06	0.06	0.187381315	0.18738133	2.070E-08	0.187381321	5.900E-09
0.08	0.08	0.248689887	0.24869004	1.518E-07	0.248689944	5.680E-08
0.1	0.1	0.309016994	0.30901770	7.031E-07	0.309017321	3.269E-07

Table 3: $\mathbf{t = \tau = 0.45}$

T	τ	Exact	Linear		Nonlinear	
			Num. Sol.	Error	Num. Sol.	Error
0	0	0	0	0	0	0
0.09	0.09	0.278991106	0.278991447	3.4130E-07	0.278991249	1.4310E-07
0.18	0.18	0.535826795	0.535862705	3.5910E-05	0.535855961	2.9166E-05
0.27	0.27	0.75011107	0.750544338	4.3327E-04	0.750612699	5.0163E-04
0.36	0.36	0.904827052	0.906705029	1.8780E-03	0.907550904	2.7239E-03
0.45	0.45	0.987688341	0.991512576	3.8242E-03	0.994225947	6.5376E-03



$t = \tau = 0.067$ $t = \tau = 0.1$ $t = \tau = 0.45$

Figure 1

The previous numerical results of Tables (1) to (3), have shown that

- It is clear the convergence of the approximate solution to the exact solution.
- The correlation between the error and the variables t and τ.
- Nonlinear type results are better than linear type in small time periods.

Application (2)

Consider the T-DQVIE

$$\phi(t,\tau) = f(t,\tau) + \ln(\phi(t,\tau)) \times \int_0^t \int_0^\tau (rt + s\tau)\gamma\big(r, s, \phi(r,s)\big)\ drds, \tag{17}$$

where in linear type $\gamma\big(r, s, \phi(r,s)\big) = \phi(s,r)$, and the nonlinear type $\gamma\big(r, s, \phi(r,s)\big) = \exp(\phi(s,r))$, we pick three-time values: T = 0.009, 0.07, 0.25 and the exact solution is $\phi(t,\tau) = (t + \tau + 1)$

Table 4: $\mathbf{t = \tau = 0.009}$

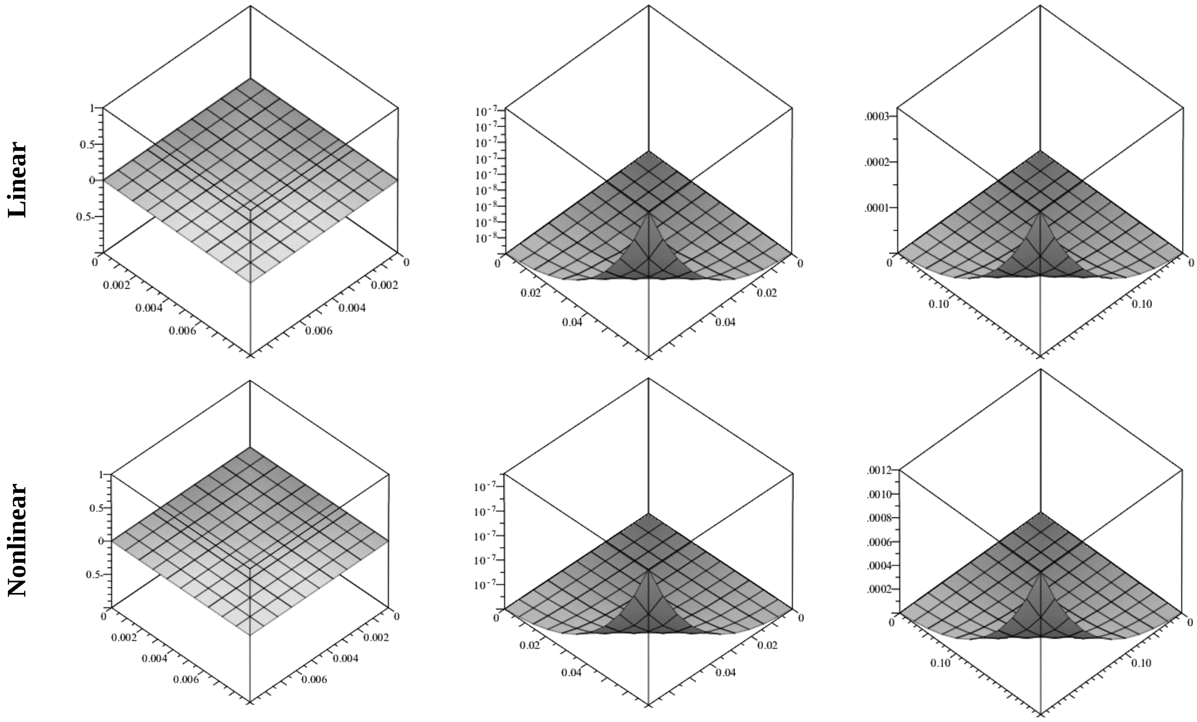
T	τ	Exact	Linear		Nonlinear	
			Num. Sol.	Error	Num. Sol.	Error
0	0	1	1	0	1	0
0.0018	0.0018	1.00360000	1.00360000	0	1.00360000	0
0.0036	0.0036	1.00720000	1.00720000	0	1.00720000	0
0.0054	0.0054	1.01080000	1.01080000	0	1.01080000	0
0.0072	0.0072	1.01440000	1.01440000	0	1.01440000	0
0.009	0.009	1.01800000	1.01800000	0	1.01800000	0

Table 5: $\mathbf{t = \tau = 0.07}$

T	τ	Exact	Linear		Nonlinear	
			Num. Sol.	Error	Num. Sol.	Error
0	0	1	1	0	1	0
0.014	0.014	1.02800	1.028000000	0	1.028000000	0
0.028	0.028	1.05600	1.056000000	0	1.056000002	2.0000E-09
0.042	0.042	1.08400	1.084000009	9.0000E-09	1.084000025	2.5000E-08
0.056	0.056	1.11200	1.112000049	4.9000E-08	1.112000144	1.4400E-07
0.07	0.07	1.14000	1.140000183	1.8300E-07	1.140000553	5.5300E-07

Table 6: $t = \tau = 0.25$

T	τ	Exact	Linear		Nonlinear	
			Num. Sol.	Error	Num. Sol.	Error
0	0	1	1	0	1	0
0.5	0.5	1.1	1.100000025	2.50000E-08	1.100000073	7.30000E-08
0.1	0.1	1.2	1.200001517	1.51700E-06	1.200004782	4.78200E-06
0.15	0.15	1.3	1.300016503	1.65030E-05	1.300055771	5.57710E-05
0.2	0.2	1.4	1.400088306	8.83060E-05	1.400317411	3.17411E-04
0.25	0.25	1.5	1.500318742	3.18742E-04	1.501201723	1.20172E-03



$t = \tau = 0.009$ $t = \tau = 0.07$ $t = \tau = 0.25$

Figure 2

The previous numerical results of Tables (4) to (6) have shown that:

- The Adomian method can be flexible to effectively solve various problems.
- The effect of the time factor is evident on the numerical solution.
- The results of the linear type are better than the nonlinear type.

6. CONCLUSIONS

By the current research, we considered a two-dimensional quadratic Volterra integral equation of the second kind with continuous kernel. The successive approximation method has been used to prove the existence and uniqueness solution. Moreover, two applications have been solved to illustrate the efficiency of the ADM. The effective effect of ADM application on quadratic integral equations is obvious.

Several cases can be established when the T-DVQIE takes special forms. For example, the T-DVQIE of the first kind and when the kernel $k(t, r; \tau, s)$ is discontinuous in the future work.

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